ISRAEL JOURNAL OF MATHEMATICS 94 (1996), 319–352

ON PHASE TRANSITIONS FOR SUBSHIFTS OF FINITE TYPE

BY

OLLE HÄGGSTRÖM

Department of Mathematics, Chalmers University of Technology
S-412 96 Gothenburg, Sweden
e-mail: olleh@math.chalmers.se

ABSTRACT

It has recently been shown that a strongly irreducible subshift of finite type in two or more dimensions may have more than one measure of maximal entropy. In this paper we obtain some results on when (i.e. for what kinds of subshifts of finite type) this happens, and when it does not. In particular, we show that the parameter of a certain subshift of finite type introduced by Burton and Steif has a critical value, below which we have a unique measure of maximal entropy, and above which we have non-uniqueness.

1. Introduction

It is well known that a strongly irreducible subshift of finite type in one dimension has a unique measure of maximal entropy (see [20]). Recently a counterexample to this in higher dimensions was shown in [3], followed by other examples in [4] and [10]. In analogy with the language of statistical mechanics, we say that a **phase transition** occurs whenever a subshift of finite type has more than one measure of maximal entropy. The main purpose of this paper is to obtain some results on when a strongly irreducible subshift of finite type exhibits a phase transition, and when it does not. A second purpose is to emphasize connections between statistical mechanics and subshifts of finite type (see [10] and [11] for more on this).

Received May 9, 1994

Let F be a finite set of at least two elements. Typically, each element of the cubic lattice $\mathbf{Z}^d$ will be assigned a value from F. x, y and z will denote elements of $\mathbf{Z}^d$, which will be called **sites**.

$\| \|_1$ will denote the L^1 norm, i.e. $\|(x_1, x_2, \dots, x_d)\|_1 = |x_1| + \dots + |x_d|$, while $\| \|_\infty$ denotes the L^∞ norm given by $\|x\|_\infty = \max_i |x_i|$. We say that y is a **nearest neighbour** of x if $\|x - y\|_1 = 1$. By the **boundary** ∂S of a finite set $S \in \mathbf{Z}^d$ we mean $\partial S = \{x \in \mathbf{Z}^d \setminus S: \exists y \in S \text{ such that } \|x - y\|_1 = 1\}$.

A **configuration** is a map $\eta: A \subseteq \mathbf{Z}^d \longrightarrow F$. We call $\eta(x)$ the **value** of the configuration at site x. Usually A will be a finite set or $\mathbf{Z}^d$ itself. A configuration $\eta: A \longrightarrow F$ is a **restriction** of a configuration $\zeta: B \longrightarrow F$ if $A \subseteq B$ and ζ agrees with η on A. We also say in this case that ζ is an **extension** of η. Note that $\mathbf{Z}^d$ acts on configurations by translation. If $y \in \mathbf{Z}^d$ set for $x \in \mathbf{Z}^d$, $T_y(x) = x + y$ and for $A \subseteq \mathbf{Z}^d$, set $T_y A = \{x + y: x \in A\}$. If $\eta: A \longrightarrow F$, we also let $T_y\eta(x) = \eta(T_{-y}(x))$ for $x \in T_y A$.

Definition 1.1: Let $\eta_i: A_i \longrightarrow F; 1 \leq i \leq K$ be a finite set S of configurations with A_i finite for each $1 \leq i \leq K$. The **subshift of finite type** (in d dimensions) corresponding to S is the set $\mathbf{X} \subseteq F^{\mathbf{Z}^d}$ consisting of all configurations $\eta: \mathbf{Z}^d \longrightarrow F$ such that for all $y \in \mathbf{Z}^d$, it is not the case that $T_y\eta$ is an extension of some η_i. (The η_i's should be thought of as the disallowed finite configurations.)

If $\mathbf{X}$ is a subshift of finite type (a SOFT), then $\mathbf{X}$ is closed in the usual product topology and is shift invariant, i.e., $\eta \in \mathbf{X}$ and $y \in \mathbf{Z}^d$ implies $T_y\eta \in \mathbf{X}$.

If $\tilde{\eta}: A \longrightarrow F$ is a configuration, we say that $\tilde{\eta}$ is **compatible** (with $\mathbf{X}$) if $\exists \eta \in \mathbf{X}$ such that $\tilde{\eta}$ is a restriction of η. Two important classes of SOFTs are given by the following definitions:

Definition 1.2: An $\mathbf{X} \subseteq F^{\mathbf{Z}^d}$ is a **nearest neighbour system** if it is a SOFT for which all the corresponding disallowed configurations $\eta_i: A_i \longrightarrow F$ have L^1-diameter 1, i.e. each A_i consists of two nearest neighbours.

Definition 1.3: Let $\mathbf{X}$ be a SOFT. $\mathbf{X}$ is **strongly irreducible** if there is an $r \geq 0$ such that whenever we have two finite compatible configurations $\eta_1: A_1 \longrightarrow F$ and $\eta_2: A_2 \longrightarrow F$ and the distance between A_1 and A_2 is greater than r, there is then an $\eta \in \mathbf{X}$ that is an extension of both η_1 and η_2.

Note that Definition 1.3 is independent of whether "distance" refers to L^1 or L^∞ norm.

A simple but nevertheless very interesting SOFT is the following:

Example 1.4: The hard-core model: Let $F = \{0,1\}$ and let $\mathbf{X}$ be the set of configurations $\eta \in F^{\mathbf{Z}^d}$ where for no two nearest neighbours $x, y \in \mathbf{Z}^d$ we have $\eta(x) = 1$, $\eta(y) = 1$, so that in other words $\mathbf{X}$ is the SOFT where the disallowed finite configurations are the d different orientations of the configuration (1 1). $\mathbf{X}$ is a nearest neighbour system, and it is also strongly irreducible since any two finite compatible configurations at distance 2 or more from each other can be embedded into a sea of 0's. This model is studied e.g. in [2]. The reason for its name is the following physical interpretation: the 0's are empty sites and the 1's are indistinguishable gas particles with non-negligible radius. In order not to overlap two particles are not allowed to occupy the same or adjacent sites.

A SOFT with a somewhat related physical interpretation is

Example 1.5: The Widom–Rowlinson model: Let $r \geq 1$ and $m \geq 3$ be integers and let $F = \{1, 2, \ldots, m\}$. Let $\mathbf{X}$ be the strongly irreducible SOFT consisting of the configurations $\eta \in F^{\mathbf{Z}^d}$ where for no $x, y \in \mathbf{Z}^d$ with $||x - y||_\infty \leq r$ we have $\eta(x) = 1$, $\eta(y) = 2$. We think of 1 and 2 as two kinds of particles that cannot coexist within distance r from each other. The name of this model refers to the work of Widom and Rowlinson [21] who introduced an analogous model of point particles in $\mathbf{R}^d$. A lattice version which closely resembles this SOFT is considered in [15].

The next definition gives a measure of the degree of complexity of a SOFT $\mathbf{X}$. Let $\Lambda_n = [-n, n]^d$ and $\mathbf{X}_n = \{\tilde{\eta}\colon \Lambda_n \longrightarrow F \text{ with } \tilde{\eta} \text{ compatible }\}$. We also let $N_n = |\mathbf{X}_n|$ ($|A|$ denotes the cardinality of A) and finally $\mathbf{X}(\tilde{\eta}) = \{\eta \in \mathbf{X}\colon \eta \text{ is an extension of } \tilde{\eta}\}$.

Definition 1.6: The **topological entropy** of $\mathbf{X}$ is

$$H(\mathbf{X}) = \lim_{n\to\infty} \frac{\log N_n}{|\Lambda_n|}.$$

Suppose that μ is a translation invariant probability measure on $\mathbf{X}$. Then the **measure theoretic entropy** of μ is

$$H(\mu) = \lim_{n\to\infty} -\frac{1}{|\Lambda_n|} \sum_{\tilde{\eta}\in\mathbf{X}_n} \mu(\mathbf{X}(\tilde{\eta})) \log \mu(\mathbf{X}(\tilde{\eta})).$$

Both of these limits exist by subadditivity. Clearly for any such μ we have $H(\mu) \leq H(\mathbf{X})$. It is in fact well known that $H(\mathbf{X}) = \sup_\mu H(\mu)$ where the supremum is taken over all translation invariant probability measures on $\mathbf{X}$.

Moreover, the supremum is achieved at some measure (see [19]). We will only be interested in probability measures in this paper, so we will simply let "measure" be short for "probability measure".

The following characterization of measures of maximal entropy for strongly irreducible SOFTs is from [3]. The "only if" direction of Proposition 1.8 was shown under some extra assumption, but it was noted in [11] that this assumption could be dropped more or less without modifying the proof.

Definition 1.7: A measure μ on $\mathbf{X} \subseteq F^{\mathbf{Z}^d}$ is said to have **uniform conditional probabilities** if for each n, the conditional distribution on Λ_n given the configuration δ on $\mathbf{Z}^d \setminus \Lambda_n$ is the uniform distribution among all those configurations which together with δ form an element of $\mathbf{X}$.

PROPOSITION 1.8: *Consider a strongly irreducible SOFT* $\mathbf{X}$. *Let* μ *be a translation invariant measure on* $\mathbf{X}$. *Then* μ *has maximal entropy if and only if it has uniform conditional probabilities.*

The reader might feel that it would be more natural in Definition 1.7 to impose uniform distributions on any finite $S \subset \mathbf{Z}^d$ and not just on the boxes Λ_n. This would in fact lead to an equivalent definition, as one can easily show using the fact that any such S is contained inside Λ_n for some sufficiently large n.

Our next example is the aforementioned SOFT from [3] showing the possibility of a phase transition.

Example 1.9: The beach model: Let $d \geq 2$, let M_1 and M_2 be positive integers such that $M_1 < M_2$, and let the alphabet be

$$F = F_1 \cup F_2 \cup F_3 \cup F_4$$

where

$$\begin{aligned} F_1 &= \{-M_2, -M_1+2, \ldots, -M_1-1\}, \\ F_2 &= \{-M_1, -M_1+1, \ldots, -1\}, \\ F_3 &= \{1, 2, \ldots, M_1\}, \\ F_4 &= \{M_1+1, M_1+2, \ldots, M_2\}. \end{aligned}$$

Call a symbol $f \in F$

$$\begin{cases} \text{negative} & \text{if } f \in F_1 \cup F_2, \\ \text{positive} & \text{if } f \in F_3 \cup F_4, \\ \text{unprivileged} & \text{if } f \in F_1 \cup F_4, \\ \text{privileged} & \text{if } f \in F_2 \cup F_3, \end{cases}$$

and consider the d-dimensional SOFT where a negative may not sit next to a positive unless they are both privileged. This SOFT is a strongly irreducible nearest neighbour system and was introduced (for the special case $M_1 = 1$) and studied by Burton and Steif [3]. For fixed d, this looks like a two-parameter family of SOFTs, but in fact the only interesting parameter (at least for our purposes) is the ratio between the number of unprivileged and the number of privileged symbols, or equivalently $M = \frac{M_2}{M_1}$. This will be explained in Section 4.

The reason why we call this model the "beach model" is the following (somewhat naive, we admit) interpretation in two dimensions: Think of $\eta(x)$ as the altitude above sea level at site x. The restrictions of the SOFT then prevents shores from being too steep.

One of the main results in [3] is that for this model with $M > 4e28^d$ we have a phase transition. We will give an alternative proof of phase transition in Section 4. An issue which is not resolved in [3] is whether the occurrence of a phase transition is increasing in M. In other words, if $m_1 < m_2$ and there is phase transition with $M = m_1$, must this be the case for $M = m_2$ as well? In the following theorem we answer this question in the affirmative.

THEOREM 1.10: *For the beach model in d dimensions, $d \geq 2$, there exists an $M_c(d) \in (1, \infty)$ such that we have a phase transition whenever $M > M_c(d)$, and a unique measure of maximal entropy whenever $M < M_c(d)$.*

Here c stands for "critical". This result is analogous to a well known result for the ferromagnetic Ising model (Theorem 2.3). We will give a lower bound of $M_c(d)$, showing that there is a unique measure of maximal entropy whenever $M < (2d^2 + d + 1)/(2d^2 + d - 1)$. Combining this with the result from [3] referred to above, we have, for $d \geq 2$,

$$\frac{2d^2 + d + 1}{2d^2 + d - 1} \leq M_c(d) \leq 4e28^d.$$

Another result from [3] is that the number of extremal measures of maximal entropy is exactly 2 when $M > 4e28^d$. A natural open question is whether this is true for all $M > M_c(d)$.

One feature of the beach model which we will obtain, and which the proof of Theorem 1.10 will be based on, is the following: Pick a configuration $\eta \in F^{\mathbf{Z}^d}$ according to some measure of maximal entropy. Then identify all positives with $+1$ and all negatives with -1. The resulting configuration $\eta' \in \{-1,1\}^{\mathbf{Z}^d}$ will turn out to be distributed as a Gibbs measure for a certain potential which we introduce in Section 2 and which somewhat resembles the Ising model.

To see the significance of strong irreducibility, it suffices to consider the following (trivial) example:

Example 1.11: Let $F = \{0,1\}$ and let $\mathbf{X}$ be the SOFT for which the disallowed configurations are those in which a 1 sits next to a 0. $\mathbf{X}$ has exactly 2 elements, and thus (even in one dimension) exactly 2 measures of maximal entropy. Of course, $\mathbf{X}$ is not strongly irreducible.

The discovery of phase transitions for the beach model and for other strongly irreducible SOFTs in 2 or more dimensions calls for general criteria for uniqueness (or non-uniqueness) of measures of maximal entropy. Our next result is a sufficient criterion for uniqueness. We first need some more definitions.

For a SOFT $\mathbf{X}$, let the **range** R of $\mathbf{X}$ be defined by

$$R(\mathbf{X}) = \max_{i\in\{1,\ldots,K\}} \max_{x,y\in A_i} ||x-y||_\infty$$

where $A_1,\ldots,A_K$ are the finite subsets of $\mathbf{Z}^d$ corresponding to the disallowed finite configurations of $\mathbf{X}$. $R(\mathbf{X})$ can be thought of as the maximal distance over which the values at sites influence each other directly. The hard-core model and the beach model have range 1, as have all nearest neighbour systems, while the Widom–Rowlinson model has range r. Let $N(\mathbf{X})$ be the cardinality of the alphabet F of $\mathbf{X}$. For any site $x \in \mathbf{Z}^d$ and any compatible configuration δ on $\mathbf{Z}^d \setminus \{x\}$ let $N_x(\delta)$ be the number of allowed values at x given δ. Note that by the shift invariance of $\mathbf{X}$ we have that $\min_\delta N_x(\delta)$ is independent of x, and define the **generosity** G of $\mathbf{X}$ by

$$G(\mathbf{X}) = \frac{\min_\delta N_x(\delta)}{N(\mathbf{X})}.$$

In words, $G(\mathbf{X}) \in (0,1]$ is the minimal fraction of the alphabet allowed at some site given the configuration on the rest of the lattice. The hard-core model has generosity $\frac{1}{2}$, the beach model has generosity $\frac{1}{2M}$ and the Widom–Rowlinson model has generosity $\frac{m-2}{m}$. Let $p_c(\mathbf{Z}^d)$ denote the critical value of independent site percolation in d dimensions (see Section 3).

THEOREM 1.12: *Let* $\mathbf{X}$ *be a SOFT in* d *dimensions with range* $R(\mathbf{X})$ *and generosity* $G(\mathbf{X})$. *If*

$$G(\mathbf{X}) > \frac{(2R(\mathbf{X})+1)^d - 2}{(2R(\mathbf{X})+1)^d - 1}$$

then $\mathbf{X}$ *has a unique measure of maximal entropy. If* $\mathbf{X}$ *is a nearest neighbour system with*

$$G(\mathbf{X}) > \frac{1}{1+p_c(\mathbf{Z}^d)}$$

then $\mathbf{X}$ *has a unique measure of maximal entropy.*

This amounts to saying that for SOFTs in a given dimension and with a given range, sufficiently large generosity implies uniqueness of measure of maximal entropy. The result is in the same spirit as Theorem 1.17 in [4], which says that if we take a nearest neighbour system $\mathbf{X}$, and add sufficiently many symbols to the alphabet, all of which are allowed to be adjacent to any other symbol including each other, then we get a unique measure of maximal entropy for the modified nearest neighbour system. In statistical mechanics these kinds of results are sometimes referred to as "high noise ergodicity criteria".

One might hope to find a bound for $G(\mathbf{X})$ yielding uniqueness uniformly in $R(\mathbf{X})$, but the next theorem says that this is not possible, except of course for the trivial bound $G(\mathbf{X}) = 1$, i.e. no forbidden configurations, in which case the measure assigning i.i.d. uniform values to all sites is (by Proposition 1.8) the unique measure of maximal entropy.

THEOREM 1.13: *In any dimension* $d \geq 2$ *and for any* $\varepsilon > 0$ *there exists a SOFT* $\mathbf{X}$ *such that* $G(\mathbf{X}) \geq 1-\varepsilon$ *and* $\mathbf{X}$ *has more than one measure of maximal entropy.*

A typical application of Theorem 1.12 is the following corollary, stating that the Widom–Rowlinson model with certain parameters has a unique measure of maximal entropy. We cannot drop the restrictions on the parameters, since for certain other parameter values phase transition occurs. In fact, the counterexample proving Theorem 1.13 will be the Widom–Rowlinson model with carefully chosen parameters.

COROLLARY 1.14: *Let* $\mathbf{X}$ *be the Widom–Rowlinson model with parameters* r *and* m. *If* $m > 2((2r+1)^d - 1)$ *then* $\mathbf{X}$ *has a unique measure of maximal entropy.*

Proof: Immediate from Theorem 1.12 using $R(\mathbf{X}) = r$ and $G(\mathbf{X}) = \frac{m-2}{m}$. ∎

Another fact which is worth mentioning here is that the concept of generosity yields a criterion for a SOFT to be irreducible. It is not hard to show that if $\mathbf{X}$ is a SOFT satisfying $G(\mathbf{X}) > \frac{1}{2}$, then $\mathbf{X}$ is strongly irreducible. Example 1.11 shows that the $>$ cannot be replaced by a $\geq$.

The rest of this paper is organized as follows. Section 2 contains the necessary prerequisites on the Ising model and other ferromagnetic models of statistical mechanics used in later sections. Section 3 quotes and discusses two uniqueness criteria for random fields, known as Dobrushin's criterion and disagreement percolation. Section 4 is devoted to the beach model; in particular, Theorem 1.10 is proved. Section 5, finally, elaborates on the Widom–Rowlinson model and proves Theorems 1.12 and 1.13.

2. The Ising model and related potentials

We consider, in this section, certain finite range potentials together with their Gibbsian random fields. Some familiarity with statistical mechanics (in particular the Ising model) is helpful, but not essential. The section is self-contained, even though the reader may want to turn to [7] or [16] for a more thorough discussion. We begin our discussion with a somewhat general setup.

A Gibbsian random field can be regarded as a certain $E^{\mathbf{Z}^d}$-valued random variable, where E is a finite set. A Gibbs measure (or Gibbs state) is the distribution of a Gibbsian random field. We will only be considering the case $E = \{-1, 1\}$. A typical element of $E^{\mathbf{Z}^d}$ will be denoted η. The potential will be given by the **interactions** $(\Phi_1, \ldots, \Phi_k) = \Phi$, which we now proceed to define. For $i = 1, \ldots, k$, let A_i be a finite nonempty subset of $\mathbf{Z}^d$ and let $\Phi_i(\eta(A_i))$ be a real-valued $\mathcal{F}_{A_i}$-measurable function (for a set $S \subseteq \mathbf{Z}^d$, $\mathcal{F}_S$ denotes the σ-algebra generated by $\eta(S)$). We let $\Phi_i(\eta(T_x A_i))$ be the corresponding $\mathcal{F}_{T_x A_i}$-valued random variable, so that if $\tilde{\eta} = T_{-x}\eta$ we have $\Phi_i(\eta(T_x A_i)) = \Phi_i(\tilde{\eta}(A_i))$, making, loosely speaking, the interactions shift invariant. The **Hamiltonian** (or the **energy**) in a box Λ_n is

$$H_n^{\Phi}(\eta) = \sum_{i=1}^{k} \sum_{\substack{x \\ T_x A_i \cap \Lambda_n \neq \emptyset}} \Phi_i(\eta(T_x A_i)),$$

i.e. $\Phi_i(\eta(T_x A_i))$ summed over all interactions and all $T_x A_i$ that intersect Λ_n.

More generally, for a finite set $S \subset \mathbf{Z}^d$, the Hamiltonian in S is defined

$$H_S^{\Phi}(\eta) = \sum_{i=1}^{k} \sum_{\substack{x \\ T_x A_i \cap S \neq \emptyset}} \Phi_i(\eta(T_x A_i)).$$

Let $r = \max_i \max_{x,y \in A_i} \|x - y\|_\infty$, so that, in two dimensions, r is the maximal height or width of any of the A_i's. For a finite set $S \subset \mathbf{Z}^d$, let the **A-boundary** $\partial_A S = \{x \in \mathbf{Z}^d \setminus S\colon \exists y \in S, i \in \{1, \ldots, k\}, z \in \mathbf{Z}^d$ so that $x, y \in T_z A_i\}$. $\partial_A S$ is clearly finite (S is contained in Λ_n for some n, and then $\partial_A S$ is contained in Λ_{n+r}), and should be thought of as being the set of sites in $\mathbf{Z}^d \setminus S$ that interact directly with S.

We also introduce the following notation: for disjoint sets $A, B \subset \mathbf{Z}^d$, and configurations ξ_1 and ξ_2 on A and B, respectively, let $\xi_1 \vee \xi_2$ be the configuration on $A \cup B$ which agrees with ξ_1 on A and with ξ_2 on B.

Let η denote a configuration on the box Λ_n. Given a configuration $\delta \in E^{\partial_A \Lambda_n}$ on the A-boundary of Λ_n, let ${}^{\Phi}\mu_n^{\delta}$ be the measure on E^{Λ_n} given by

$${}^{\Phi}\mu_n^{\delta}(\eta) = \frac{e^{-H_n^{\Phi}(\delta \vee \eta)}}{{}^{\Phi}Z_n^{\delta}}$$

where

$${}^{\Phi}Z_n^{\delta} = \sum_{\eta \in E^{\Lambda_n}} e^{-H_n^{\Phi}(\delta \vee \eta)}$$

making ${}^{\Phi}\mu_n^{\delta}$ a probability measure. It is common practice in statistical mechanics to include a factor $\frac{1}{T}$, the inverse temperature, in the exponent. We shall, however, consider the temperature to be fixed; we can then incorporate the factor $\frac{1}{T}$ into the Hamiltonian. This amounts to setting $T = 1$.

Definition 2.1: A measure μ on $E^{\mathbf{Z}^d}$ is called a **Gibbs state** for Φ if for each n, the conditional distribution on Λ_n given the configuration δ on $\mathbf{Z}^d \setminus \Lambda_n$ is given by ${}^{\Phi}\mu_n^{\delta'}$ above, where δ' is the restriction of δ to $\partial_A \Lambda_n$.

The corresponding statement where Λ_n is replaced by an arbitrary finite set $S \subset \mathbf{Z}^d$ follows easily; compare with the comment following Proposition 1.8. One can also show, for arbitrary Φ, the existence of at least one Gibbs state. There may sometimes be more than one Gibbs state, in which case a phase transition is said to occur. When this happens and when it doesn't is an issue which has been studied extensively for three decades, and which can be said to be the statistical mechanics analogue of the more recent question of phase transitions for subshifts of finite type.

The above is the general setup for finite range potentials. We now turn to the examples which will be of use to us in Section 4. The first one is the well known Ising model, while we propose to call the second, which seems to be new, the site-centered ferromagnet. We choose this name because the model is ferromagnetic in the sense of Definition 2.6 below, and because, in contrast to the Ising model, it is the sites themselves, rather than pairs of nearest neighbours, that can be either "satisfied" (low energy) or "unsatisfied" (high energy). The reason for introducing this new model is its close relationship with the beach model which we will establish in Section 4.

Example 2.2: The Ising model: We define the 2-dimensional standard Ising model with zero external field, the generalization to higher dimensions being obvious. Let $d = 2$, $E = \{-1, 1\}$, $k = 2$, and the A_i's be given by

$$A_1 = \{(0,0),(0,1)\},$$
$$A_2 = \{(0,0),(1,0)\}.$$

This means that there are only nearest neighbour interactions. Fix a constant J and let the Φ_i's be given by

$$\Phi_1\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) = \Phi_1\left(\begin{smallmatrix}-1\\-1\end{smallmatrix}\right) = \Phi_2(1\ 1) = \Phi_2(-1\ \ -1) = -J,$$
$$\Phi_1\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right) = \Phi_1\left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right) = \Phi_2(-1\ 1) = \Phi_2(1\ \ -1) = J,$$

where e.g. $\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)$ denotes the configuration η on A_1 given by $\eta(0,0) = 1$, $\eta(0,1) = -1$. The sites are thought of as atoms, with spins pointing either up $(+1)$ or down (-1). Note that the model is symmetric with respect to $\{-1, 1\}$. J is called the coupling constant. When $J > 0$, nearest neighbours will tend to have the same value, and we speak of the ferromagnetic Ising model, while the case $J < 0$ is referred to as the antiferromagnetic Ising model. Our interest will be focused upon the ferromagnetic case in which we think of a pair of nearest neighbours as being unsatisfied if they have different values and satisfied otherwise. One of the classical results in statistical mechanics is that for sufficiently large J, there are (at least) two different extremal Gibbs states. The following theorem (see [7], [16]) tells us this and more:

THEOREM 2.3: *For the Ising model with $d \geq 2$ there exists $J_c(d) \in (0, \infty)$ such that in d dimensions there is a unique Gibbs state for $0 \leq J < J_c(d)$ while for $J > J_c(d)$ there is more than one Gibbs state.*

The exact value of $J_c(d)$ is only known in 2 dimensions: $J_c(2) = \frac{1}{2}\log(1+\sqrt{2})$. Some bounds are known in higher dimensions. One of these is that $J_c(d)$ is decreasing in d. When $d = 1$ the Ising model reduces to an irreducible aperiodic two state Markov chain for which a phase transition never occurs.

Example 2.4: The site-centered ferromagnet: For any dimension d, let $E = \{-1, 1\}$, $k = 1$, and let A_1 consist of the origin together with its $2d$ nearest neighbours. Given a configuration $\eta \in E^{\mathbf{Z}^d}$ we say that a site x is **satisfied** if the value at x equals the value at each of its nearest neighbours; otherwise we say x is **unsatisfied**. Let L be a positive real number and let

$$\Phi_1(\eta(A_1)) = \begin{cases} 0 & \text{if the origin is satisfied,} \\ L & \text{if the origin is unsatisfied.} \end{cases}$$

This model is, just like the Ising model, symmetric with respect to $\{-1, 1\}$. In Section 4 we will prove the following theorem which tells us that the site-centered ferromagnet has more than this in common with the Ising model.

THEOREM 2.5: *For the site-centered ferromagnet with $d \geq 2$ there exists $L_c(d) \in (0, \infty)$ such that in d dimensions there is a unique Gibbs state for $0 \leq L < L_c(d)$ while for $L > L_c(d)$ there is more than one Gibbs state.*

There are differences, however, between the Ising model and the site-centered ferromagnet. Here is one such difference: The Ising model is Markov, in the sense that the conditional distribution on a box Λ_n given a configuration on $\mathbf{Z}^d \setminus \Lambda_n$ depends only on the values on $\partial\Lambda_n$. The corresponding conditional distribution for the site-centered ferromagnet depends on the values on the larger set $\partial_A\Lambda_n = \{x \in \mathbf{Z}^d \setminus \Lambda_n : \exists y \in \Lambda_n \text{ such that } \|x - y\|_1 \leq 2\}$.

Now it is time to make precise what is meant in general by "ferromagnetism".

Definition 2.6: A potential $\Phi = (\Phi_1, \ldots, \Phi_k)$ on $\{-1, 1\}^{\mathbf{Z}^d}$ is said to be **ferromagnetic** if for $i = 1, \ldots, k$,

$$\Phi_i(\eta(A_i)) = -J_i \prod_{x \in A_i} \eta(x) + K_i$$

where $J_1, \ldots, J_k$ are nonnegative real numbers and $K_1, \ldots, K_k$ are real numbers.

The K_i's are in fact irrelevant since they have no influence on the Gibbs measures corresponding to the potentials, but it will turn out in Example 2.9 that it is convenient to allow for the term.

One reason why ferromagnetism is interesting is the next result on ferromagnetic measures on finite sets. It is one of the so-called Griffiths inequalities, discussed in Section IV.1 in [16]. Let S be a finite set, let $A_1, \dots, A_k$ be given subsets of S, and let $J_1, \dots, J_k$ and $K_1, \dots, K_k$ be as in Definition 2.6. Let the measure μ_S on $\{-1,1\}^S$ be given by

$$\mu_S(\eta) = \frac{e^{-H_S(\eta)}}{Z_S}$$

where

$$H_S(\eta) = -\sum_{i=1}^{k} J_i \prod_{x \in A_i} \eta(x) + K_i$$

and Z_S is a normalizing constant. The K_i's are still irrelevant as they have no effect on μ_S.

PROPOSITION 2.7: GRIFFITHS' INEQUALITY: *If S and μ_S are as above, then for any $x \in S$ and any $i \in \{1, \dots, k\}$*

$$\frac{\partial}{\partial J_i} \mu_S(\{\eta\colon \eta(x) = 1\}) \geq 0.$$

This inequality can be used to derive results on ferromagnetic potentials on $\{-1,1\}^{\mathbf{Z}^d}$ by considering certain limits as S increases to $\mathbf{Z}^d$. We will do so in Section 4.

Clearly, the Ising model with $J \geq 0$ is ferromagnetic in the sense of Definition 2.6. It is somewhat less obvious that the site-centered ferromagnet is ferromagnetic in this sense. To see this, we first need the following lemma:

LEMMA 2.8: *Let $Y_n = \{-1,1\}^n$ where $n \geq 2$. Let the function $\phi_n\colon Y_n \longrightarrow \mathbf{Z}$ be given by*

$$\phi_n(y_1, \dots, y_n) = \sum_{B \in \mathbf{B}_n} \prod_{i \in B} y_i,$$

where $\mathbf{B}_n$ is the set of all subsets $B \subseteq \{1, \dots, n\}$ (including the empty set) which have even cardinality. Then

$$\phi_n(y) = \begin{cases} 2^{n-1} & \text{if } y_1 = \cdots = y_n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Suppose first that $y_1 = \cdots = y_n$. Then $\prod_{i\in B} y_i = 1$ for all $B \in \mathbf{B}_n$, and we have $\phi_n(y) = |\mathbf{B}_n| = 2^{n-1}$. On the other hand, suppose that it is not the case that $y_1 = \cdots = y_n$. Then we can find $j, k \in \{1, \ldots, n\}$ such that $y_j = -y_k$. Now define a relation $\sim$ on $\mathbf{B}_n$ by

$$B_1 \sim B_2 \text{ iff } \begin{cases} \text{for } i = j, k, \ i \text{ is an element of } B_1 \cup B_2 \text{ but not of } B_1 \cap B_2, \\ \text{for } i \neq j, k, \ i \text{ is an element of neither or both of } B_1 \text{ and } B_2. \end{cases}$$

It is easy to see that $\sim$ satisfies

$$\begin{cases} B_1 \sim B_2 \Rightarrow B_2 \sim B_1 \text{ and } B_2 \neq B_1 \\ B_1 \sim B_2, \ B_1 \sim B_3 \Rightarrow B_2 = B_3 \\ \text{for all } B_1 \text{ there is a } B_2 \text{ such that } B_1 \sim B_2 \end{cases}$$

so that it divides $\mathbf{B}_n$ into pairs. Also

$$\prod_{i\in B_1} y_i = -\prod_{i\in B_2} y_i$$

whenever $B_1 \sim B_2$, whence everything cancels in the sum $\sum_{B\in\mathbf{B}_n} \prod_{i\in B} y_i$ so that $\phi_n(y) = 0$. ∎

Now it is easy to verify that the following ferromagnetic potential is nothing but an alternative description of the site-centered ferromagnet with parameter L.

Example 2.9: The site-centered ferromagnet revisited: For any dimension d let $k = 2^{2d} - 1$. Let $A \subset \mathbf{Z}^d$ consist of the origin together with its $2d$ nearest neighbours. Let $A_1, \ldots, A_k$ be the k nonempty subsets of A whose cardinalities are even, and let for $i = 1, \ldots, k$

$$\Phi_i(\eta(A_i)) = \frac{L}{2^{2d}}(1 - \prod_{x\in A_i} \eta(x)).$$

By Lemma 2.8 we now get, in the terminology of Example 2.4,

$$\sum_{i=1}^{k} \Phi_i(\eta(A_i)) = \begin{cases} 0 & \text{if the origin is satisfied,} \\ L & \text{otherwise.} \end{cases}$$

This establishes that the site-centered ferromagnet really is ferromagnetic.

3. Two uniqueness criteria

In this section we discuss two uniqueness criteria for random fields with given conditional probabilities. The first one is the celebrated Dobrushin's criterion, which has been used in very many contexts ever since it was introduced in [6]. The second criterion, which we refer to as disagreement percolation, is much more recent [1]. Both criteria say that if the conditional distribution of the value at a single site depends sufficiently little on its surroundings, then we have no phase transition. The exact mathematical formulation of this is somewhat different for the two criteria, and it turns out that sometimes one of them is more effective, sometimes the other. The reader will find examples of this in Sections 4 and 5.

Let F be a finite set and let X_1 and X_2 be two F-valued random variables with distributions μ_1 and μ_2, respectively. The **variational distance** between X_1 and X_2 is defined as

$$d(X_1, X_2) = \frac{1}{2} \sum_{f \in F} |\mu_1(f) - \mu_2(f)|$$

and we write $d(\mu_1, \mu_2)$ for the same quantity. An equivalent definition would be

$$d(X_1, X_2) = \max_{E \subseteq F} |\mu_1(E) - \mu_2(E)|.$$

Let P be a measure on $F^{\mathbf{Z}^d}$. Let P_δ^x denote the conditional distribution under P of the value at x given the configuration δ on $\mathbf{Z}^d \setminus \{x\}$. Suppose this quantity is independent of x (more precisely, suppose that for any y we have $P_\delta^x = P_{T_y\delta}^{T_y x}$). The measures for SOFTs satisfying the equivalent conditions in Proposition 1.8, as well as the Gibbs states considered in Section 2, have this property. Let

$$\rho(y, x) = \sup_\delta \sup_\eta d(P_\delta^x, P_\eta^x)$$

where the supremum is taken over all configurations δ on $\mathbf{Z}^d \setminus \{x\}$ and all configurations η such that $\eta = \delta$ everywhere except at the site y. Loosely speaking, $\rho(y, x)$ is the maximal influence which the single site y can have on the distribution at x. Under a certain technical condition called "quasilocality", which we do not need to bother about in this paper, we have the following result; see [6] and [7] for more details.

Proposition 3.1: Dobrushin's criterion: *If*

$$\sum_{y \in Z^d \setminus \{x\}} \rho(y, x) < 1,$$

then P is the only measure on $F^{\mathbf{Z}^d}$ with the given conditional distributions.

For the next uniqueness result, we need to say a few words about percolation. A standard reference on percolation is [9]; see also [14] for a brief introduction.

Let $\mathbf{G}$ be a graph whose vertex set is $\mathbf{Z}^d$ (so that "sites" and "vertices" from now on are synonyms) and whose edge set is shift invariant in the sense that the presence of an edge between x and y depends only on $x-y$. The graph is assumed to be connected and locally finite, i.e. the number of edges incident to a given vertex is finite. By the shift invariance this number is the same for all vertices; we denote it $N_{\mathbf{G}}$. Two vertices are said to be **adjacent** if they are incident to the same edge. An important special case is the **nearest neighbour graph** on $\mathbf{Z}^d$, where two sites are adjacent if and only if they are nearest neighbours. By a **path** we mean a finite or infinite sequence of distinct sites $x_1, x_2, \ldots$ in which consecutive sites are adjacent.

In **site percolation** each site is assigned a value 0 or 1 according to some measure μ on $\{0,1\}^{\mathbf{Z}^d}$. A site whose value is 1 (0) is called **open** (**closed**). The main objects of study in percolation are **open paths**, by which we mean paths on which all vertices are open. Particularly interesting is the possible existence of an infinite open path. **Bond percolation**, which we will not deal with here, is similar, except that in that model it is the edges rather than the vertices that are open or closed.

For $p \in [0,1]$, let μ_p denote the measure under which each site independently is open with probability p and closed with probability $1-p$. The **critical probability** for site percolation on $\mathbf{G}$ is defined by

$$p_c(\mathbf{G}) = \inf\{p \in [0,1]\colon \mu_p(\text{there exists an infinite open path}) > 0\}.$$

When $d \geq 2$, $p_c(\mathbf{G})$ is in fact nontrivial, i.e. $p_c \in (0,1)$. We let, with some abuse of notation, $p_c(\mathbf{Z}^d)$ denote the critical value for site percolation on the nearest neighbour graph on $\mathbf{Z}^d$. The critical value $p_c(\mathbf{G})$ is usually extremely hard to calculate exactly, but numerous bounds are known. The disagreement percolation uniqueness criterion relies on lower bounds for p_c. The best lower bound today of $p_c(\mathbf{Z}^2)$ is

$$p_c(\mathbf{Z}^2) > 0.5416$$

by a computer-assisted proof in [18]. Another bound, which will be of use to us later, is the following, from [12]:

PROPOSITION 3.2: *The critical probability for site percolation on a graph* $\mathbf{G}$ *as above satisfies*

$$P_c(\mathbf{G}) \geq \frac{1}{N_{\mathbf{G}} - 1}.$$

We now return to the measure P on $F^{\mathbf{Z}^d}$. Suppose P has finite range, i.e. that there is a finite set $B \subset \mathbf{Z}^d$ such that the conditional distribution P_δ^x depends only on the configuration on $T_x B$. Suppose furthermore that B is the smallest set with this property, so that

$$T_x B = \{y\colon \rho(y, x) > 0\}.$$

Let $\mathbf{G}$ be the graph with vertex set $\mathbf{Z}^d$ and for which there is an edge between x and y if and only if $y \in B$. It is easy to check that $\rho(x, y) = 0$ whenever $\rho(y, x) = 0$ so that this definition of $\mathbf{G}$ is consistent. The uniqueness criterion from [1] now says

PROPOSITION 3.3: DISAGREEMENT PERCOLATION: *If*

$$\sup_{\delta, \eta \in F^{\mathbf{Z}^d \setminus \{x\}}} d(P_\delta^x, P_\eta^x) < p_c(\mathbf{G})$$

then P *is the only measure on* $F^{\mathbf{Z}^d}$ *with the given conditional distributions.*

The proof of this result and reason for the name "disagreement percolation" is roughly as follows: The proof is based on a coupling (see [17], [16], [1]) between two processes with measures P and P' with the prescribed conditional distributions. The coupling is constructed in such a way that the set of sites where the two processes take on different values (disagree) is stochastically dominated by a set of open sites under the measure μ_p for some $p < p_c(\mathbf{G})$. Therefore there is a.s. no infinite path of disagreement (i.e. no infinite path of distinct vertices all of which disagree for the two processes), and any finite set $S \subset \mathbf{Z}^d$ must be surrounded by some (random) layer on which the processes agree. The distribution inside this layer turns out to be the same for the two processes, and the result follows.

4. Results for the beach model

A key idea in our proof of Theorem 1.10 is to establish an equivalence between the beach model with parameter M and the site-centered ferromagnet with parameter $L = \log M$, so that the proof can be reduced to proving Theorem 2.5. The equivalence, which is similar in spirit to results in [10] and [11], amounts to the following:

Let the SOFT $\mathbf{X}$ be given by the beach model with parameter M. Consider a measure μ_b, with uniform conditional probabilities, for $\mathbf{X}$. If all positives in F are identified with $+1$ and all negatives with -1, then the measure on $\{-1,1\}^{\mathbf{Z}^d}$ induced by μ_b turns out to be a Gibbs state for the site-centered ferromagnet with parameter L.

Going the other way is also possible. Take a Gibbs measure μ_s for the site-centered ferromagnet. Given a realization $\eta \in \{-1,1\}^{\mathbf{Z}^d}$, assign values from F to each site independently in such a way that a positive satisfied site gets its value from the uniform distribution on $\{1,\ldots,M_2\}$, a positive unsatisfied site gets its value from the uniform distribution on $\{1,\ldots,M_1\}$, and similarly for negative sites. This procedure yields an element of $\mathbf{X}$ which is distributed according to a measure with uniform conditional probabilities. (This can of course only be done if e^L is rational, since $e^L = M = \frac{M_2}{M_1}$.)

An important feature of these mappings between measures for the two different models is that they constitute a bijection. Hence, (non-)uniqueness of measures with uniform conditional probabilities for $\mathbf{X}$ is equivalent to (non-)uniqueness of Gibbs states for the corresponding site-centered ferromagnet.

We now turn to proving this equivalence. Fix M_1 and M_2 and the dimension d. Let $\Omega_0 = \{-1,1\}^{\mathbf{Z}^d}$, $\Omega_1 = \{1,\ldots,M_1\}^{\mathbf{Z}^d}$, and $\Omega_2 = \{1,\ldots,M_2\}^{\mathbf{Z}^d}$. We will construct a measure on the product space $\Omega = \Omega_0 \times \Omega_1 \times \Omega_2$. An element of Ω will be denoted $\omega = (\omega_0,\omega_1,\omega_2)$. Let μ_0 be an arbitrary Gibbs measure for the site-centered ferromagnet with parameter $L = \log M = \log \frac{M_2}{M_1}$, and let μ_1 and μ_2 be uniform i.i.d. measures on Ω_1 and Ω_2, respectively. Let μ be the product measure $\mu_0 \times \mu_1 \times \mu_2$.

Let $\mathbf{X}$ be the beach SOFT corresponding to M_1 and M_2, with symbol set $F = \{-M_2,\ldots,-2,-1,1,2,\ldots,M_2\}$. Let Y be the random element of $F^{\mathbf{Z}^d}$ constructed from Ω, under the measure μ, defined by

$$Y(x) = Y(x,\omega) = \begin{cases} \omega_0(x)\omega_1(x) & \text{if } x \text{ is unsatisfied} \\ \omega_0(x)\omega_2(x) & \text{otherwise} \end{cases}$$

where x being unsatisfied means that $\omega_0(x) \neq \omega_0(y)$ for some nearest neighbour y of x. It is easy to check that Y must be an element of $\mathbf{X}$. Moreover, we have

LEMMA 4.1: *Y has uniform conditional probabilities with respect to $\mathbf{X}$.*

The following notation is useful for the proof, and also later. For $S \subset \mathbf{Z}^d$ and a configuration $\eta \in F^S$, let $\mathrm{sgn}(\eta)$ denote pointwise sgn, i.e. $\mathrm{sgn}(\eta)(x) = \frac{\eta(x)}{|\eta(x)|}$ for all $x \in S$. Note that $\mathrm{sgn}(Y(x)) = \omega_0(x)$.

Proof: Fix configurations ξ' and ξ'', on $\partial\Lambda_n$ and $\mathbf{Z}^d \setminus (\Lambda_n \cup \partial\Lambda_n)$, respectively, in such a way that $\xi' \vee \xi''$ is compatible. Let ξ be a configuration on Λ_n such that $\xi \vee \xi' \vee \xi''$ forms an element of $\mathbf{X}$. What we need to show is that

$$\mu\left(Y(\Lambda_n) = \xi \middle| Y(\mathbf{Z}^d \setminus \Lambda_n) = \xi' \vee \xi''\right)$$

is independent of the choice of ξ. We first compute a different, but related, conditional probability.

Given $\xi \vee \xi' \vee \xi''$, let j denote the number of unsatisfied sites in $\Lambda_n \cup \partial\Lambda_n$, and let ${}^L Z_n^{\xi' \vee \xi''}$ be the normalizing constant for the distribution on Λ_n of the site-centered ferromagnet with parameter L and boundary condition $\mathrm{sgn}(\xi' \vee \xi'')$. We have

$$\begin{aligned}
&\mu\left(Y(\Lambda_n \cup \partial\Lambda_n) = \xi \vee \xi' \middle| Y(\mathbf{Z}^d \setminus (\Lambda_n \cup \partial\Lambda_n)) = \xi'', \mathrm{sgn}(Y(\partial\Lambda_n)) = \mathrm{sgn}(\xi')\right) \\
&= \mu\left(Y(\Lambda_n \cup \partial\Lambda_n) = \xi \vee \xi' \middle| \mathrm{sgn}(Y(\mathbf{Z}^d)) = \mathrm{sgn}(\xi \vee \xi' \vee \xi'')\right) \\
&\quad \mu\left(\mathrm{sgn}(Y(\Lambda_n)) = \mathrm{sgn}(\xi) \middle| \mathrm{sgn}(Y(\mathbf{Z}^d \setminus \Lambda_n)) = \mathrm{sgn}(\xi' \vee \xi'')\right) \\
&= M_1^{-j} M_2^{-(|\Lambda_n \cup \partial\Lambda_n| - j)} \frac{e^{-Lj}}{{}^L Z_n^{\xi' \vee \xi''}} \\
&= \left(\frac{M_2}{M_1}\right)^j M_2^{-|\Lambda_n \cup \partial\Lambda_n|} \frac{e^{-Lj}}{{}^L Z_n^{\xi' \vee \xi''}} \\
&= e^{Lj} M_2^{-|\Lambda_n \cup \partial\Lambda_n|} \frac{e^{-Lj}}{{}^L Z_n^{\xi' \vee \xi''}} \\
&= \frac{M_2^{-|\Lambda_n \cup \partial\Lambda_n|}}{{}^L Z_n^{\xi' \vee \xi''}}
\end{aligned}$$

which only depends on the things we condition on in the first line of the computation. Hence, this conditional distribution is uniform over all possible configurations. If we now condition further, this time on $Y(\partial\Lambda_n)$, the obtained conditional distribution is of course still uniform, i.e.

$$\begin{aligned}
&\mu\left(Y(\Lambda_n \cup \partial\Lambda_n) = \xi \vee \xi' \middle| Y(\mathbf{Z}^d \setminus \Lambda_n) = \xi' \vee \xi''\right) \\
&= \mu\left(Y(\Lambda_n) = \xi \middle| Y(\mathbf{Z}^d \setminus \Lambda_n) = \xi' \vee \xi''\right)
\end{aligned}$$

is uniform, and we are done. ∎

So now we know that any Gibbs state for the site-centered ferromagnet yields a measure with uniform conditional probabilities for the beach model. The next lemma tells us how to go the other way.

LEMMA 4.2: *Let Y' be a random element of $\mathbf{X}$ satisfying the uniform conditional probabilities property. Then* $\operatorname{sgn}(Y')$ *is distributed as a Gibbs state for the site-centered ferromagnet with parameter L.*

Proof: The proof will be by comparing Y' to Y, where Y is the $\mathbf{X}$-valued random variable constructed above, and for which we have that $\operatorname{sgn}(Y)$ is distributed according to the Gibbs measure μ_0.

Let $A|B \stackrel{d}{=} C|D$ be short for the statement "the conditional distribution of A given B is the same as the conditional distribution of C given D", where A, B, C and D are random variables. We are done if we can show that for any n,

$$\operatorname{sgn}(Y'(\Lambda_n))\Big|\operatorname{sgn}(Y'(\mathbf{Z}^d\setminus\Lambda_n)) \stackrel{d}{=} \operatorname{sgn}(Y(\Lambda_n))\Big|\operatorname{sgn}(Y(\mathbf{Z}^d\setminus\Lambda_n)).$$

We proceed to prove this. We have

$$Y'(\Lambda_{n+1})\Big|Y'(\mathbf{Z}^d\setminus\Lambda_{n+1}) \stackrel{d}{=} Y(\Lambda_{n+1})\Big|Y(\mathbf{Z}^d\setminus\Lambda_{n+1})$$

since both conditional probabilities are uniform over all allowed configurations. If we now condition on $\operatorname{sgn}(Y(\Lambda_{n+1}\setminus\Lambda_n))$ as well, the conditional distributions must still be equal, i.e.

$$\begin{aligned} &Y'(\Lambda_{n+1})\Big|Y'(\mathbf{Z}^d\setminus\Lambda_{n+1}),\operatorname{sgn}(Y'(\Lambda_{n+1}\setminus\Lambda_n)) \\ \stackrel{d}{=}\ &Y(\Lambda_{n+1})\Big|Y(\mathbf{Z}^d\setminus\Lambda_{n+1}),\operatorname{sgn}(Y(\Lambda_{n+1}\setminus\Lambda_n)). \end{aligned}$$

Since $\operatorname{sgn}(Y'(\Lambda_n))$ and $\operatorname{sgn}(Y(\Lambda_n))$ can be expressed in terms of $Y'(\Lambda_{n+1})$ and $Y(\Lambda_{n+1})$, respectively, it follows that

$$\begin{aligned} &\operatorname{sgn}(Y'(\Lambda_n))\Big|Y'(\mathbf{Z}^d\setminus\Lambda_{n+1}),\operatorname{sgn}(Y'(\Lambda_{n+1}\setminus\Lambda_n)) \\ \stackrel{d}{=}\ &\operatorname{sgn}(Y(\Lambda_n))\Big|Y(\mathbf{Z}^d\setminus\Lambda_{n+1}),\operatorname{sgn}(Y(\Lambda_{n+1}\setminus\Lambda_n)). \end{aligned}$$

It follows from the construction preceding Lemma 4.1 that the right hand side only can depend on $Y(\mathbf{Z}^d\setminus\Lambda_{n+1})$ through $\operatorname{sgn}(Y(\mathbf{Z}^d\setminus\Lambda_{n+1}))$, because for $x\in\mathbf{Z}^d\setminus\Lambda_{n+1}$ we have that T_xA_1 (with A_1 defined as in Example 2.4) does not intersect Λ_n. The same thing must then hold for the left hand side as well. Hence, $Y'(\mathbf{Z}^d \setminus \Lambda_{n+1})$ and $Y(\mathbf{Z}^d \setminus \Lambda_{n+1})$ can be replaced by $\text{sgn}(Y'(\mathbf{Z}^d \setminus \Lambda_{n+1}))$ and $\text{sgn}(Y(\mathbf{Z}^d \setminus \Lambda_{n+1}))$, respectively, in the conditioning, and we get

$$\text{sgn}(Y'(\Lambda_n))\Big|\,\text{sgn}(Y'(\mathbf{Z}^d \setminus \Lambda_n)) \stackrel{d}{=} \text{sgn}(Y(\Lambda_n))\Big|\,\text{sgn}(Y(\mathbf{Z}^d \setminus \Lambda_n))$$

so the proof is complete. ∎

Using Lemmae 4.1 and 4.2 we can now go both ways: from a Gibbs state for the site-centered ferromagnet to a measure with uniform conditional probabilities on $\mathbf{X}$, and the other way around. That this actually induces a one-to-one correspondence between the Gibbs states and the measures with uniform conditional probabilities follows once we have

(i) when going from the site-centered ferromagnet to the beach model and back again, we end up with the Gibbs state we started with, and

(ii) when going from the beach model to the site-centered ferromagnet and back to the beach model, we get back the measure with which we started.

(i) is immediate while (ii) is a consequence of the following property of measures with uniform conditional probabilities for the beach model: conditional on the signs on $\mathbf{Z}^d$, all values are independent with, for each x, $\eta(x)$ being uniform over all values that are allowed given the signs at x and its nearest neighbours. This follows easily from the fact that $\eta(x)$, conditional on $\eta(\mathbf{Z}^d \setminus \{x\})$, is uniformly distributed over all allowed values.

It is now easy to explain why M is the only interesting parameter for the beach model (although one can very well realize this without considering the relation with the site-centered ferromagnet). Let $\mathbf{X}$ and $\mathbf{X}'$ be two beach SOFTs such that $\frac{M_2}{M_1} = \frac{M_2'}{M_1'}$ (with the obvious notation). Then these correspond to the same site-centered ferromagnet, whence their measures with uniform conditional probabilities yield the same distributions of positives of negatives. We feel that, at least in this context, this is where the "essence" of the model lies, since what remains given the signs is just independent uniform random variables. In particular, $\mathbf{X}$ has a unique measure of maximal entropy if and only if $\mathbf{X}'$ has.

We now concentrate our efforts on deriving properties for the site-centered ferromagnet. The corresponding properties for the beach model will follow easily using the above correspondence.

First, we discuss a certain monotonicity property for the site-centered ferromagnet. For a nice and general discussion on this kind of monotonicity arguments, see Sections II.2 and III.2 in [16]. For $S \subseteq \mathbf{Z}^d$ and two configurations $\delta, \eta \in \{-1,1\}^S$ we write $\delta \preceq \eta$ if $\delta(x) \leq \eta(x)$ for all $x \in S$.

Definition 4.3: Let μ and ν be probability measures on $\{-1,1\}^S$. We say that $\mu \preceq \nu$ if there exists a probability measure m on $\{-1,1\}^S \times \{-1,1\}^S$ whose first and second marginals are μ and ν respectively (i.e. a coupling of μ and ν) and such that

$$m\{(\delta,\eta): \delta \preceq \eta\} = 1.$$

An equivalent definition would be the following: $\mu \preceq \nu$ if for every function $f: \{-1,1\}^S \to \mathbf{R}$ which is continuous in the usual product topology, and monotone in the sense that $f(\delta) \leq f(\eta)$ whenever $\delta \preceq \eta$, we have

$$\int f d\mu \leq \int f d\nu.$$

For a configuration $\delta \in \{-1,1\}^{\mathbf{Z}^d \setminus \{x\}}$, let μ_δ^L denote the conditional distribution of the value at x given δ, for the site-centered ferromagnet with parameter L.

LEMMA 4.4: *Let $\delta, \eta \in \{-1,1\}^{\mathbf{Z}^d \setminus \{x\}}$ be such that $\delta \preceq \eta$. Then*

$$\mu_\delta^L \preceq \mu_\eta^L.$$

Proof: Let the set A_x consist of the site x together with its $2d$ nearest neighbours. Let n_δ^+ denote the number of unsatisfied sites in A_x for the configuration on $\mathbf{Z}^d$ which equals δ on $\mathbf{Z}^d \setminus \{x\}$ and which equals $+1$ at x, and let n_δ^-, n_η^+ and n_η^- be the obvious analogous quantities. We have

$$\mu_\delta^L(\{+1\}) = \frac{e^{-Ln_\delta^+}}{e^{-Ln_\delta^+} + e^{-Ln_\delta^-}} = \frac{1}{1 + e^{L(n_\delta^+ - n_\delta^-)}}$$

and

$$\mu_\eta^L(\{+1\}) = \frac{e^{-Ln_\eta^+}}{e^{-Ln_\eta^+} + e^{-Ln_\eta^-}} = \frac{1}{1 + e^{L(n_\eta^+ - n_\eta^-)}}.$$

Some thought reveals that $n_\delta^+ - n_\delta^- \geq n_\eta^+ - n_\eta^-$ whence $\mu_\delta^L(\{+1\}) \leq \mu_\eta^L(\{+1\})$ so that $\mu_\delta^L \preceq \mu_\eta^L$. ∎

Now that we have Lemma 4.4, it should not be hard to believe that the corresponding statement where the single site x is replaced by a finite set S holds as well. The next lemma tells us that this is so. For a finite set $S \subset \mathbf{Z}^d$ and a configuration $\delta \in \{-1,1\}^{\mathbf{Z}^d \setminus S}$ let $\mu^L_{S,\delta}$ denote the conditional distribution given δ of the configuration on S for the site-centered ferromagnet with parameter L.

LEMMA 4.5: *Let $S \subset \mathbf{Z}^d$ be finite and let $\delta, \eta \in \{-1,1\}^{\mathbf{Z}^d \setminus S}$ be such that $\delta \preceq \eta$. Then*

$$\mu^L_{S,\delta} \preceq \mu^L_{S,\eta}.$$

Before giving the proof we remark that the same thing holds true for all ferromagnetic potentials with pairwise interactions only (see [16]), while for interactions involving three or more sites simultaneously, as is the case here, the result does not hold in general (neither does Lemma 4.4). Moreover, the result cannot be extended to infinite S, as the reader may convince herself once we have established a phase transition.

Proof of Lemma 4.5: The proof is somewhat standard (see e.g. the proof of Lemma 2.3 in [3]) so we will be a bit brief.

We first define a continuous finite state Markov process P_δ on $\{-1,1\}^S$ which is irreducible and has stationary distribution $\mu^L_{S,\delta}$. Let δ' denote an element of $\{-1,1\}^S$. The dynamics are given by flip rates as follows. Each site x flips (changes its value) independently of all other sites, at rate

$$c_\delta(x,\delta') = \begin{cases} e^{-n^-_{\delta\vee\delta'}} & \text{if } \delta'(x) = 1 \\ e^{-n^+_{\delta\vee\delta'}} & \text{if } \delta'(x) = -1 \end{cases}$$

where $n^-_{\delta\vee\delta'}$ and $n^+_{\delta\vee\delta'}$ are defined as in the proof of Lemma 4.4.

It is not hard to show that P_δ is irreducible. To show that $\mu^L_{S,\delta}$ is a stationary distribution, it suffices to show that it is reversible, which means that for any $\delta'_1, \delta'_2 \in \{-1,1\}^S$ we have

$$\mu^L_{S,\delta}(\delta'_1)q(\delta'_1,\delta'_2) = \mu^L_{S,\delta}(\delta'_2)q(\delta'_2,\delta'_1)$$

where $q(\delta'_1,\delta'_2)$ is the rate at which δ'_1 switches to δ'_2. To see this, first note that $q(\delta'_1,\delta'_2) = 0$ unless δ'_1 and δ'_2 differ at exactly one site. So suppose $\delta'_1 = \delta'_2 = \delta'$ on $S \setminus \{x\}$ and $\delta'_1(x) = 1$, $\delta'_2(x) = -1$. Then

$$\frac{q(\delta'_1,\delta'_2)}{q(\delta'_2,\delta'_1)} = \frac{c_\delta(x,\delta'_1)}{c_\delta(x,\delta'_2)} = \frac{e^{-n^-_{\delta\vee\delta'}}}{e^{-n^+_{\delta\vee\delta'}}} = \frac{\mu^L_{S,\delta}(\delta'_2)}{\mu^L_{S,\delta}(\delta'_1)}$$

whence $\mu^L_{S,\delta}$ is reversible and stationary.

Now define the Markov process P_η with stationary distribution $\mu^L_{S,\eta}$ in the same way. In order to show $\mu^L_{S,\delta} \preceq \mu^L_{S,\eta}$ we construct a coupling for P_δ and P_η, i.e. a Markov process on $\{-1,1\}^S \times \{-1,1\}^S$ whose first and second marginals are P_δ and P_η. For $(\delta',\eta') \in \{-1,1\}^S \times \{-1,1\}^S$ the site x flips according to

$$
\begin{array}{llll}
(1,1) & \text{flips to} \left\{\begin{array}{l}(-1,1)\\(1,-1)\\(-1,-1)\end{array}\right. & \text{at rate} & \left\{\begin{array}{l}c_\delta(x,\delta') - \min(c_\delta(x,\delta'),c_\eta(x,\eta'))\\ c_\eta(x,\eta') - \min(c_\delta(x,\delta'),c_\eta(x,\eta'))\\ \min(c_\delta(x,\delta'),c_\eta(x,\eta'))\end{array}\right. \\
(-1,-1) & \text{flips to} \left\{\begin{array}{l}(1,-1)\\(-1,1)\\(1,1)\end{array}\right. & \text{at rate} & \left\{\begin{array}{l}c_\delta(x,\delta') - \min(c_\delta(x,\delta'),c_\eta(x,\eta'))\\ c_\eta(x,\eta') - \min(c_\delta(x,\delta'),c_\eta(x,\eta'))\\ \min(c_\delta(x,\delta'),c_\eta(x,\eta'))\end{array}\right. \\
(1,-1) & \text{flips to} \left\{\begin{array}{l}(-1,-1)\\(1,1)\end{array}\right. & \text{at rate} & \left\{\begin{array}{l}c_\delta(x,\delta')\\ c_\eta(x,\eta')\end{array}\right. \\
(-1,1) & \text{flips to} \left\{\begin{array}{l}(1,1)\\(-1,-1)\end{array}\right. & \text{at rate} & \left\{\begin{array}{l}c_\delta(x,\delta')\\ c_\eta(x,\eta').\end{array}\right.
\end{array}
$$

This is the so-called Vasershtein (or basic) coupling discussed e.g. in [17] and [16]. It is easy to check that the marginal flip rates are the desired ones. Moreover, one can use the proof of Lemma 4.4 to see that the flip rates are such that the set $H = \{(\delta',\eta') \in \{-1,1\}^S \times \{-1,1\}^S : \delta' \preceq \eta'\}$ is invariant for the coupled process; this is where $\delta \preceq \eta$ is used. Starting in a state $(\delta',\eta') \in H$ the process stays in H forever. By general Markov chain theory the distribution approaches some stationary limit measure ν as $t \to \infty$. Clearly $\nu(H) = 1$ and the two marginals of ν are $\mu^L_{S,\delta}$ and $\mu^L_{S,\eta}$, whence $\mu^L_{S,\delta} \preceq \mu^L_{S,\eta}$. ∎

Let $\mu^L_{n,+}$ be the conditional distribution of the configuration on Λ_n given the configuration which equals +1 all over $\mathbf{Z}^d \setminus \Lambda_n$. So $\mu^L_{n,+}$ is a measure on $\{-1,1\}^{\Lambda_n}$ but it can equally well be thought of as a measure on $\{-1,1\}^{\mathbf{Z}^d}$ which is concentrated on the event that every site in $\mathbf{Z}^d \setminus \Lambda_n$ has the value +1. With this interpretation in mind, note that Lemma 4.5 implies that

$$\mu^L_{n_2,+} \preceq \mu^L_{n_1,+} \quad \text{for } n_1 \le n_2.$$

Monotonicity and compactness now guarantee that the measure

$$\mu^L_+ = \lim_{n\to\infty} \mu^L_{n,+}$$

exists. Clearly, μ^L_+ is a Gibbs state for the site-centered ferromagnet. Moreover, we have, for an arbitrary Gibbs state ν for the same model and for any n,

$$\nu \preceq \mu^L_{n,+}$$

whence

$$\nu \preceq \mu_+^L.$$

We refer to this property by saying that μ_+^L is a **maximal** Gibbs state. It is easy to see that μ_+^L must be translation invariant. By symmetry there must of course also exist a **minimal** Gibbs state μ_-^L obtained similarly. The following lemma is useful:

LEMMA 4.6: *The following three statements concerning the site-centered ferromagnet with parameter L are equivalent:*

(i) $\mu_+^L = \mu_-^L$.

(ii) *There is only one Gibbs state.*

(iii) *There is only one translation invariant Gibbs state.*

Proof: (i) $\Rightarrow$ (ii) since $\mu \preceq \nu$ and $\nu \preceq \mu$ imply $\mu = \nu$, while (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) are immediate. ∎

Note also that the symmetry between μ_+^L and μ_-^L implies that

$$\mu_+^L(\{\eta\colon \eta(x) = 1\}) + \mu_-^L(\{\eta\colon \eta(x) = 1\}) = 1$$

whence

$$\mu_+^L(\{\eta\colon \eta(x) = 1\}) \geq \frac{1}{2}.$$

We now state three lemmae whose proofs we defer slightly. Together they will yield Theorem 2.5.

LEMMA 4.7: *Suppose there is a unique Gibbs state for the site-centered ferromagnet in d dimensions with parameter L_1. Then the same thing holds for the site-centered ferromagnet in d dimensions with parameter L_2 whenever $L_2 \leq L_1$.*

LEMMA 4.8: *The site-centered ferromagnet in $d \geq 2$ dimensions with parameter L has more than one Gibbs state if*

$$L > 2^{2d-2}\log(1+\sqrt{2}).$$

LEMMA 4.9: *The site-centered ferromagnet in d dimensions with parameter L has a unique Gibbs state if*

$$L < \log\left(\frac{2d^2+d+1}{2d^2+d-1}\right).$$

Proof of Theorem 2.5: Lemma 4.7 implies that there is an $L_c(d) \in [0, \infty]$ such that there is a unique Gibbs state when $L < L_c(d)$ and more than one Gibbs state when $L > L_c(d)$. Lemma 4.8 implies

$$L_c(d) \le 2^{2d-2} \log\left(1+\sqrt{2}\right) < \infty$$

and Lemma 4.9 implies

$$L_c(d) \ge \log\left(\frac{2d^2+d+1}{2d^2+d-1}\right) > 0$$

so that $L_c(d) \in (0, \infty)$. ∎

Proof of Theorem 1.10: Recall the one-to-one correspondence between Gibbs states for the site-centered ferromagnet and measures with uniform conditional probabilities for the beach model, and suppose $L > L_c(d)$. By Lemma 4.6, the site-centered ferromagnet with parameter L has at least two translation invariant Gibbs states. The beach model with parameter $M = e^L$ then has at least two translation invariant measures with uniform conditional probabilities. Hence it has, by Proposition 1.8, at least two measures of maximal entropy. Suppose on the other hand that $L < L_c(d)$. Then the site-centered ferromagnet has a unique Gibbs state, so that the beach model with parameter $M = e^L$ must have a unique measure of maximal entropy. Since e^L is increasing in L the result follows with $M_c(d) = e^{L_c(d)}$. ∎

The bound

$$M_c(d) \ge \frac{2d^2+d+1}{2d^2+d-1}$$

mentioned in the introduction is now an immediate consequence of Lemma 4.9.

Strictly speaking we did not have to use Lemma 4.8 in order to derive Theorems 2.5 and 1.10, since we could instead have used the result from [3] that there are at least two measures of maximal entropy for the beach model whenever $M > 4e28^d$. We include Lemma 4.8 anyway since our proof involves a technique quite different from the one in [3].

We now complete this section by proving Lemmae 4.7, 4.8 and 4.9. First, however, we quote a result from [16] which we will need.

LEMMA 4.10: *Let μ and ν be measures on $\{-1,1\}^{\mathbf{Z}^d}$ such that $\mu \preceq \nu$. If*

$$\mu(\{\eta\colon \eta(x)=1\}) = \nu(\{\eta\colon \eta(x)=1\})$$

for all $x \in \mathbf{Z}^d$, then $\mu = \nu$.

Proof of Lemma 4.7: Consider the measure $\mu_{n,+}^{L_1}$ and view it as a measure on $\{-1,1\}^{\Lambda_n}$. By the parameterization in Example 2.9, the interactions describing this measure fits into the framework of Griffiths' inequality. It is easy to check that we do not get any boundary problems, the point being that for interactions Φ_i involving sites both in Λ_n and in $\mathbf{Z}^d \setminus \Lambda_n$ we can forget about the latter since (due to the boundary condition which equals $+1$ all over $\mathbf{Z}^d \setminus \Lambda_n$) their contribution to Φ_i is just a multiplication with $+1$. Griffiths' inequality now yields, for $L_2 \leq L_1$ and any $x \in \Lambda_n$,

$$\mu_{n,+}^{L_2}(\{\eta: \eta(x) = 1\}) \leq \mu_{n,+}^{L_1}(\{\eta: \eta(x) = 1\})$$

so that by letting $n \to \infty$ we have

$$\mu_{+}^{L_2}(\{\eta: \eta(x) = 1\}) \leq \mu_{+}^{L_1}(\{\eta: \eta(x) = 1\}).$$

Suppose now that $\mu_+^{L_1}$ is the only Gibbs state with parameter L_1. Then $\mu_+^{L_1} = \mu_-^{L_1}$ so that by symmetry

$$\mu_{+}^{L_1}(\{\eta: \eta(x) = 1\}) = \frac{1}{2}$$

whence

$$\mu_{+}^{L_2}(\{\eta: \eta(x) = 1\}) = \frac{1}{2}.$$

Again, symmetry yields

$$\mu_{-}^{L_2}(\{\eta: \eta(x) = 1\}) = \frac{1}{2}$$

so that by Lemma 4.10 we have $\mu_+^{L_2} = \mu_-^{L_2}$ and by Lemma 4.6 this is the only Gibbs state with parameter L_2. ∎

Proof of Lemma 4.8: Using the parameterization of the site-centered ferromagnet in Example 2.9, a similar application of Griffiths' inequality as in the previous proof shows that the Ising model with coupling constant J has a unique Gibbs state whenever the site-centered ferromagnet with parameter $L = 2^{2d-1}J$ has a unique Gibbs state. But since for $d \geq 2$ the Ising model with coupling constant $J > \frac{1}{2}\log(1+\sqrt{2})$ has a phase transition the result follows. ∎

Proof of Lemma 4.9: The idea of this proof is to apply Dobrushin's criterion to the site-centered ferromagnet. It is possible to use disagreement percolation instead if one prefers to, but that will yield a worse lower bound for $L_c(d)$.

Given a configuration $\delta \in \{-1,1\}^{\mathbf{Z}^d \setminus \{x\}}$, let A_x, n_δ^+ and n_δ^- be as in the proof of Lemma 4.4. The distribution μ_δ^L of the value at x depends only on the values on the set

$$B_x = \{y \in \mathbf{Z}^d \setminus \{x\} \colon \exists z \in \mathbf{Z}^d \text{ such that } \{x,y\} \subseteq T_z A_x\}.$$

We think of B_x as the set of sites which interact directly with x. B_x can be written as the union of the disjoint sets B_x^1, B_x^2 and B_x^3, where

$$\begin{aligned} B_x^1 &= \{y \in \mathbf{Z}^d \colon \|y-x\|_1 = 1\}, \\ B_x^2 &= \{y \in \mathbf{Z}^d \colon \|y-x\|_1 = 2, \|y-x\|_\infty = 1\}, \text{ and} \\ B_x^3 &= \{y \in \mathbf{Z}^d \colon \|y-x\|_1 = 2, \|y-x\|_\infty = 2\}. \end{aligned}$$

This partition of B_x describes the three different positions, in relation to x and modulo rotations and reflections, which a $y \in B_x$ can have. Let $\delta_y \in \{-1,1\}^{\mathbf{Z}^d \setminus \{x\}}$ be the configuration obtained from δ by flipping the value at y. Furthermore, let $\Delta n_\delta = n_\delta^+ - n_\delta^-$. We have

$$d(\mu_\delta^L, \mu_{\delta_y}^L) = |\mu_\delta^L(\{+1\}) - \mu_{\delta_y}^L(\{+1\})| = \left| \frac{1}{1+e^{L\Delta n_\delta}} - \frac{1}{1+e^{L\Delta n_{\delta_y}}} \right|.$$

Since

$$|\{z \in \mathbf{Z}^d \colon \{x,y\} \subseteq T_z A_x\}| = \begin{cases} 2 & \text{for } y \in B_x^1 \cup B_x^2 \\ 1 & \text{for } y \in B_x^3 \end{cases}$$

it is easy to verify that

$$|\Delta n_\delta - \Delta n_{\delta_y}| \le \begin{cases} 2 & \text{for } y \in B_x^1 \cup B_x^2 \\ 1 & \text{for } y \in B_x^3 \end{cases}$$

for all δ, with equality for some δ (depending on y). Since the function

$$f(t) = \frac{1}{1+e^t}$$

is decreasing and has a derivative $f'(t)$ such that for all t, $f'(t) = f'(-t)$ and $f'(t) \ge f'(0)$, it follows that

$$\rho(y,x) = \sup_{\delta \in \{-1,1\}^{Z^d \setminus \{x\}}} d(\mu_\delta^L, \mu_{\delta_y}^L) \le \begin{cases} \frac{1}{1+e^{-L}} - \frac{1}{1+e^L} & \text{for } y \in B_x^1 \cup B_x^2, \\ \frac{1}{1+e^{-L}} - \frac{1}{2} & \text{for } y \in B_x^3. \end{cases}$$

Again, it is not hard to check, by finding suitable δ, that this inequality is in fact an equality. Since $|B_x^1 \cup B_x^2| = 2d^2$ and $|B_x^3| = 2d$, the sum in Dobrushin's criterion becomes

$$\begin{aligned}\sum_{y\in Z^d\setminus\{x\}}\rho(y,x) &= \sum_{y\in B_x}\rho(y,x)\\ &= 2d^2\left(\frac{1}{1+e^{-L}}-\frac{1}{1+e^{L}}\right)+2d\left(\frac{1}{1+e^{-L}}-\frac{1}{2}\right)\\ &= (2d^2+d)\frac{e^L+1}{e^L-1}\end{aligned}$$

which is less than 1 when

$$e^L < \frac{2d^2+d+1}{2d^2+d-1},$$

i.e. when

$$L < \log\left(\frac{2d^2+d+1}{2d^2+d-1}\right),$$

confirming our result. ∎

5. Proofs of Theorems 1.12 and 1.13

Theorem 1.12 will be proved using disagreement percolation. It would certainly be reasonable to try to prove it using Dobrushin's criterion, but, in contrast to the proof of Lemma 4.9, this will here yield uniqueness of measure of maximal entropy for a smaller set of SOFTs. See [1] for further comparisons for the two criteria.

Proof of Theorem 1.12: Suppose $\mathbf{X}$ has generosity G and alphabet F with cardinality N. Let $\delta,\eta \in F^{\mathbf{Z}^d\setminus\{x\}}$ be two compatible configurations. Let $F_\delta \in F$ denote the set of allowed values at x given δ, and define F_η analogously. We may without loss of generality assume $|F_\delta| \le |F_\eta|$, and we have

$$G \le \frac{|F_\delta|}{N} \le \frac{|F_\eta|}{N}.$$

Also,

$$\frac{|F_\eta\setminus F_\delta|}{N} \le 1-\frac{|F_\delta|}{N} \le 1-G.$$

The conditional distributions P_δ^x and P_η^x of the value at x are uniform on F_δ and F_η, respectively. It is easy to see that

$$d(P_\delta^x,P_\eta^x) = \frac{|F_\eta\setminus F_\delta|}{|F_\eta|} \le \frac{1-G}{G}$$

whence

$$\sup_{\delta,\eta\in Z^d\setminus\{x\}} d(P_\delta^x, P_\eta^x) \le \frac{1-G}{G}$$

since δ and η are arbitrary.

If $\mathbf{X}$ is a nearest neighbour system we have that P_δ^x and P_η^x only depend on nearest neighbours of x, so that by disagreement percolation we have uniqueness if

$$\frac{1-G}{G} < p_c(\mathbf{Z}^d)$$

and the second part of the theorem follows.

To get the first part, note that if $\mathbf{X}$ has range R, then P_δ^x and P_η^x only depend on the values at sites within L^∞-distance R from x. Consider the graph $\mathbf{G}$ with vertex set $\mathbf{Z}^d$ and where there is an edge connecting x and y iff $\|x-y\|_\infty \le R$. Each vertex is incident to $(2R+1)^d - 1$ edges, so that by Proposition 3.2 we have that the critical probability $p_c(\mathbf{G})$ for site percolation on $\mathbf{G}$ satisfies

$$p_c(\mathbf{G}) \ge \frac{1}{(2R+1)^d - 2}.$$

Disagreement percolation now yields uniqueness whenever

$$\frac{1-G}{G} < \frac{1}{(2R+1)^d - 2}$$

and the first part of the theorem follows. ∎

The key to proving Theorem 1.13 is the following lemma, the proof of which most of our remaining work will be devoted to:

LEMMA 5.1: *Consider the Widom–Rowlinson model in 2 dimensions with parameters r and m. For any fixed m there is an R such that the model has more than one measure of maximal entropy as long as $r > R$.*

We remark that this result is true for any dimension $d \ge 2$, as the reader may be convinced by studying our proof and comparing with [15]. We prefer to stay in 2 dimensions to keep things simpler, as this turns out to be sufficient to prove Theorem 1.13.

Proof of Theorem 1.13: For $d = 2$ the result is almost immediate from Lemma 5.1. Given $\varepsilon > 0$, let $\mathbf{X}$ be the Widom–Rowlinson model with $m \ge \frac{2}{\varepsilon}$ so that

$$G(\mathbf{X}) = \frac{m-2}{m} \ge 1 - \varepsilon$$

and pick r sufficiently large to guarantee more than one measure of maximal entropy.

We now turn to the case $d = 3$. Let $\mathbf{X}_=$ be the 3-dimensional SOFT with the same symbol set as $\mathbf{X}$ and with rules as follows. On each horizontal plane the rules are the same as those for $\mathbf{X}$, while in the vertical direction there are no restrictions. Clearly, $G(\mathbf{X}_=) = G(\mathbf{X})$. Pick two different measures of maximal entropy μ_1 and μ_2 for $\mathbf{X}$, and construct measures ν_1 and ν_2 on $\mathbf{X}_=$ given by

$$\nu_1 = \prod_{i \in \mathbf{Z}} \mu_1$$

and

$$\nu_2 = \prod_{i \in \mathbf{Z}} \mu_2$$

so that in words, under ν_1, all horizontal planes are independent with each plane being given the measure μ_1, and similarly for ν_2. It is easy to check that ν_1 and ν_2 are measures of maximal entropy for $\mathbf{X}$ using Proposition 1.8. Finally for $d \geq 4$ we can proceed as for $d = 3$ in the obvious way, so the proof is complete. ∎

Proof of Lemma 5.1: The following terminology, as well as parts of the proof, is borrowed from [15]. For fixed m, r and a configuration $\delta \in \{1, \ldots, m\}^S$, where $S \subseteq \mathbf{Z}^d$, we say that a site $x \in S$ is

$$\begin{cases} \text{black} & \text{if } \delta(x) = 1, \\ \text{grey} & \text{if } \delta(x) \neq 1 \text{ but } \exists y \text{ such that } \|x - y\|_\infty \leq r \text{ and } \delta(y) = 1, \\ \text{red} & \text{if } \delta(x) = 2, \\ \text{pink} & \text{if } \delta(x) \neq 2 \text{ but } \exists y \text{ such that } \|x - y\|_\infty \leq r \text{ and } \delta(y) = 2, \\ \text{white} & \text{if none of the above apply.} \end{cases}$$

Note that a site can be both grey and pink at the same time. We write "reddish" for "red or pink", and "blackish" for "black or grey".

Suppose now that there is a unique measure of maximal entropy μ for the Widom–Rowlinson model with parameters m and r. The symmetry with respect to $\{1, 2\}$ of the model implies that

$$\mu(x \text{ reddish}) = \mu(x \text{ blackish}) \geq \frac{1}{2}(1 - \mu(x \text{ white})).$$

We have, by Proposition 1.8, that

$$\mu(x \text{ white}) = \mu(x \text{ white} | W_x)\mu(W_x) \leq \mu(x \text{ white} | W_x) = \frac{m-2}{m}$$

where W_x is the event that there is no black or red site within L^∞-distance r from x (except possibly at x). Hence

$$(*) \qquad \mu(x \text{ reddish}) \geq \frac{1}{2}\left(1 - \frac{m-2}{m}\right) = \frac{1}{m}.$$

We are done if we can contradict $(*)$ by picking r sufficiently large and constructing a suitable measure of maximal entropy. This will be accomplished by a so-called contour (or Peierls) argument (see [7], [15]).

Let $\mu_{n,1}^{m,r}$ be the uniform distribution over all allowed configurations on Λ_n given the configuration on $\mathbf{Z}^2 \setminus \Lambda_n$ where every site has the value 1.

Let Λ_n^* be the nearest neighbour graph on

$$\left\{-n-\frac{1}{2}, -n+\frac{1}{2}, \dots, n+\frac{1}{2}\right\} \times \left\{-n-\frac{1}{2}, -n+\frac{1}{2}, \dots, n+\frac{1}{2}\right\}.$$

This is sometimes called the dual graph for the nearest neighbour graph on Λ_n. A **circuit** C in Λ_n^* is a sequence of distinct (except for the first and the last) vertices $v_1, v_2, \dots, v_k, v_1$ such that consecutive vertices are nearest neighbours. C can be identified with the edge sequence $e_1, e_2, \dots, e_k$ where for $i = 1, \dots, k-1$, e_i is the edge between v_i and v_{i+1}, and e_k is the edge between v_k and v_1. As one travels along a circuit clockwise, each edge has one well-defined site $y \in \mathbf{Z}^2$ immediately to the left, and one immediately to the right; the former being outside of the circuit and the latter inside. Given $\delta \in \{1, \dots, m\}^{\Lambda_n}$, a circuit C is said to be a **contour** for $x \in \Lambda_n$ (and δ) if

(a) C surrounds x, and

(b) as one travels around C clockwise, all sites immediately to the left are non-reddish, and all sites immediately to the right are reddish.

With the value 1 all over $\mathbf{Z}^2 \setminus \Lambda_n$, so that $\mathbf{Z}^2 \setminus \Lambda_n$ is all black, it is clear (if not, see [7]) that if $x \in \Lambda_n$ is reddish there must exist at least one contour for x. We will now show that for large r the probability of a contour for x is small.

For a contour C_x for x, let $B_1 \subseteq \Lambda_n$ denote the set of sites in Λ_n that are located outside of C_x. Furthermore, let $B_2 \subseteq \Lambda_n$ be the set of sites which are located inside C_x but at L^∞-distance at most r from some site located outside C_x. Finally, let $B_3 \subseteq \Lambda_n$ be the set of sites inside C_x which are not members of B_2. B_1, B_2 and B_3 form a partition of Λ_n.

Some thought reveals that for each bond of C_x travelled clockwise, the r nearest sites looking straight to the right belong to B_2, and each site in B_2 can be related in this way to at most 3 edges in C_x, whence

$$|B_2| \geq \frac{r}{3}|C_x|,$$

where $|C_x|$ denotes the length of C_x. Also, all sites in B_2 must be pink.

We now estimate the probability that a fixed circuit C_x that surrounds x is a contour for x. Fix $\delta \in \{1,\ldots,m\}^{\Lambda_n}$ so that C_x is a contour. Let δ_1, δ_2 and δ_3 be the restrictions of δ to B_1, B_2 and B_3, respectively. Let the transformation $T_{C_x}\colon \{1,\ldots,m\}^{B_1\cup B_3} \to \{1,\ldots,m\}^{B_1\cup B_3}$ be defined by flipping all 1's on B_3 to 2 and all 2's on B_3 to 1, and keeping all other values as they are. Note that T_{C_x} is a bijection.

We now compare the probability of $\delta_1\vee\delta_3$ to that of $T_{C_x}(\delta_1\vee\delta_3)$. Since each site in B_2 can take the values $3, 4, \ldots, m$ given $\delta_1\vee\delta_3$ and the event $\{C_x \text{ is a contour}\}$, while they can take the values $1, 3, 4, \ldots, m$ given $T_{C_x}(\delta_1\vee\delta_3)$, we have

$$\frac{\mu_{n,1}^{m,r}(\delta_1\vee\delta_3,\ C_x \text{ is a contour})}{\mu_{n,1}^{m,r}(T_{C_x}(\delta_1\vee\delta_3))} = \left(\frac{m-2}{m-1}\right)^{|B_2|} \leq \left(\frac{m-2}{m-1}\right)^{\frac{r}{3}|C_x|}.$$

Summing over all $\delta_1\vee\delta_3 \in \{1,\ldots,m\}^{B_1\cup B_3}$ that can make C_x a contour for x we get

$$\begin{aligned}\mu_{n,1}^{m,r}(C_x \text{ is a contour}) &= \sum_{\delta_1\vee\delta_3}\mu_{n,1}^{m,r}(\delta_1\vee\delta_3,\ C_x \text{ is a contour}) \\ &\leq \frac{\sum_{\delta_1\vee\delta_3}\mu_{n,1}^{m,r}(\delta_1\vee\delta_3,\ C_x \text{ is a contour})}{\sum_{\delta_1\vee\delta_3}\mu_{n,1}^{m,r}(T_{C_x}(\delta_1\vee\delta_3))} \\ &\leq \left(\frac{m-2}{m-1}\right)^{\frac{r}{3}|C_x|}.\end{aligned}$$

A simple argument (see e.g. [7], Lemma 6.13) shows that there are at most $l3^{l-1}$ circuits of length l that surround x. Hence the expected number $E^{m,r}$ of contours for x satisfies

$$E^{m,r} \leq \sum_{l=1}^{\infty} l3^{l-1}\left(\frac{m-2}{m-1}\right)^{\frac{r}{3}l}$$

uniformly in the box size n. Since $(\frac{m-2}{m-1})^{\frac{r}{3}}$ can be made arbitrarily small by making r large, it is clear that for large r the sum converges and can even be made as small as one prefers. Hence

$$\mu_{n,1}^{m,r}(x \text{ reddish}) < \frac{1}{2m}$$

uniformly in x and n for some sufficiently large r. Compactness guarantees the existence of the measure

$$\mu_1^{m,r} = \lim_{i\to\infty} \mu_{n_i,1}^{m,r}$$

for some subsequence $(i_1, i_2, \ldots)$ of $(1, 2, \ldots)$. This measure on $\{1, 2, \ldots, m\}^{\mathbf{Z}^2}$ has uniform conditional probabilities with respect to the Widom–Rowlinson model with parameters m and r, and satisfies

$$\mu_1^{m,r}(x \text{ reddish}) \leq \frac{1}{2m} < \frac{1}{m}$$

for all x. Monotonicity arguments similar to those in Section 4 show that $\mu_1^{m,r}$ is translation invariant. Hence it is a measure of maximal entropy and we have a contradiction to $(*)$, as desired. ∎

ACKNOWLEDGEMENT: I am grateful to my thesis advisor Jeff Steif for his constant support throughout the preparation of this paper.

References

[1] J. van den Berg and C. Maes, *Disagreement percolation in the study of Markov fields*, The Annals of Probability **22** (1994), 749–763.

[2] J. van den Berg and J. E. Steif, *Percolation and the hard core lattice gas model*, Stochastic Processes and their Applications **49** (1994), 179–197.

[3] R. Burton and J. E. Steif, *Nonuniqueness of measures of maximal entropy for subshifts of finite type*, Ergodic Theory and Dynamical Systems **14** (1994), 213–236.

[4] R. Burton and J. E. Steif, *New results on measures of maximal entropy*, Israel Journal of Mathematics **89** (1995), 275–300.

[5] R. Burton and J. E. Steif, *Quite weak Bernoulli with exponential rate and percolation for random fields*, to appear in Stochastic Processes and their Applications.

[6] P. L. Dobrushin, *The description of a random field by means of conditional probabilities and conditions of its regularity*, Theory of Probability and its Applications **13** (1968), 197–224.

[7] H. Georgii, *Gibbs Measures and Phase Transitions*, de Gruyter, New York, 1988.

[8] R. B. Griffiths, *Correlations in Ising ferromagnets*, Journal of Mathematical Physics **8** (1967), 478–489.

[9] G. Grimmett, *Percolation*, Springer, New York, 1989.

[10] O. Häggström, *A subshift of finite type that is equivalent to the Ising model*, to appear in Ergodic Theory and Dynamical Systems.

[11] O. Häggström, *On the relation between finite range potentials and subshifts of finite type*, Probability Theory and Related Fields **101** (1995), 469–478.

[12] J. M. Hammersley, *Percolation processes. Lower bounds for the critical probability*, Annals of Mathematical Statistics **28** (1957), 790–795.

[13] D. G. Kelly and S. Sherman, *General Griffiths inequalities on correlations in Ising ferromagnets*, Journal of Mathematical Physics **9** (1968), 466–484.

[14] H. Kesten, *Percolation theory and first-passage percolation*, The Annals of Probability **15** (1987), 1231–1271.

[15] J. L. Lebowitz and G. Gallavotti, *Phase transitions in binary lattice gases*, Journal of Mathematical Physics **12** (1971), 1129–1133.

[16] T. M. Liggett, *Interacting Particle Systems*, Springer, New York, 1985.

[17] T. Lindvall, *Lectures on the Coupling Method*, Wiley, New York, 1992.

[18] M. V. Menshikov and K. D. Pelikh, *Percolation with several types of defects. Estimation of the critical probability on the square lattice*, Matematicheskie Zametski **46** (1989), 38–47 (in Russian).

[19] M. Misiurewicz, *A short proof of the variational principle for a* $\mathbf{Z}_+^N$*-action on a compact space*, Astérisque **40** (1975), 147–157.

[20] W. Parry, *Intrinsic Markov chains*, Transactions of the American Mathematical Society **112** (1964), 55–66.

[21] B. Widom and J. S. Rowlinson, *New model for the study of liquid–vapor phase transitions*, Journal of Chemical Physics **52** (1970), 1670–1684.